J Math Chem (2010) 47:1177–1183
DOI 10.1007/s10910-009-9613-6

BRIEF COMMUNICATION

Nonconvex polyhedra by repeated truncation of semiregular polyhedra

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Received: 21 September 2009 / Accepted: 27 September 2009 / Published online: 14 October 2009

Abstract Some of the semiregular (Archimedean) polyhedra (**1–13** in Table 1) afford on truncation polyhedra that contain vertices where the sum of planar degrees for the faces which meet at those vertices is equal to (for **17, 18**, and **23** in Table 3) or higher than 360° (**21, 22, 24–26** in Table 3). Therefore such polyhedra are nonconvex.

Keywords Semiregular (Archimedean) polyhedra · Truncation · Nonconvex

1 Introduction

The five regular polyhedra and the 13 semiregular polyhedra are vertex-transitive. Among the 13 Archimedean (semiregular) polyhedra (15 if the enantiomerism of the two snub polyhedra is taken into account), seven have vertices of degree 3, four have degree 4, and the two snub polyhedra have degree 5. The truncation of a polyhedron [1] replaces a vertex of degree *d* by a regular *d*-gon and doubles the number of vertices for every face of the polyhedron. All vertices of truncated polyhedra are trivalent, i.e. have degree 3.

For instance, the truncated tetrahedron is a semiregular polyhedron that has four hexagons derived from the four triangular faces of the tetrahedron, and six triangles derived from the six vertices of the tetrahedron (Fig. 1). Each of the 12 vertices is trivalent and lies at the meeting place of two hexagons and a triangle.

The most famous chemical structure derived from a semiregular polyhedron is C_{60}, buckminsterfullerene, with the structure of a truncated icosahedron, which was discovered accidentally by mass spectrometric techniques [2]. After C_{60} was obtained in sufficient amounts for detailed spectrophotometric and NMR techniques [3], the

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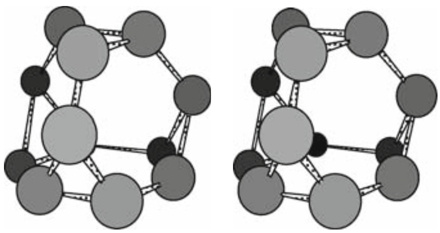

Fig. 1 Stereo-view of the truncated tetrahedron (**1**)

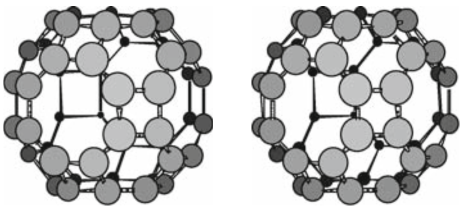

Fig. 2 Stereo-view of the truncated cuboctahedron (great rhombicuboctahedron), **8**

discovery was honored with the Nobel Prize for Chemistry in 1996 [4–6]. Many reviews and monographs have appeared in the few years since this discovery [7–23].

Among the semiregular polyhedra there are two pairs related by truncation, namely **8** is the truncated **2**, and **13** is the truncated **7**. The truncated cuboctahedron (great rhombicuboctahedron, **8**) is a semiregular polyhedron that has six octagons (derived from the six squares of the cuboctahedron **2**), eight hexagons (derived from the eight triangles of the cuboctahedron, and twelve squares or tetragons (derived from the 12 vertices of the cuboctahedron), as seen in Fig. 2. Each of the 48 vertices is trivalent, and lies at the meeting place of an octagon, a hexagon, and a square.

Similarly, the truncated icosidodecahedron (great rhombicosidodecahedron, **13**) is a semiregular polyhedron that has twelve decagons (derived from the twelve pentagons of the icosidodecahedron **7**), twenty hexagons (derived from the twenty triangles of the icosidodecahedron, and thirty squares (derived from the 30 vertices of the icosidodecahedron).

A few useful relationships are recalled. The angle α between two sides of any regular n-gon (Table 1) is:

$$\alpha = \pi\ (n-2)\ /n(\text{in radians}) = 180^\circ(n-2)/n(\text{in degrees}) \qquad (1)$$

Euler's formulas for polyhedra (with various n_i-gonal faces F_i and various vertrices V_j with degree d_j) are:

$$V + F = E + 2 \qquad (2)$$

$$2E = \Sigma_i n_i F_i = \Sigma_j d_j V_j \qquad (3)$$

For the 13 semiregular polyhedra, Table 2 displays their names, symbols (with the numbers of n-gon faces (F_i) meeting at a vertex indicated as superscripts for n, namely n_i^{Fi}), number V of vertices, number E of edges, and total number of faces $F = \Sigma_i F_i$),

Table 1 Angles α of n-gons

Polygon n	Angle α (degrees)
3	60
4	90
5	108
6	120
8	135
10	144
12	150
16	157.5
20	162

Table 2 Semiregular (Archimedean) polyhedra and some of their numerical data

No.	Semiregular polyhedron	Faces meeting at a vertex	Symbol	d	V	E	F	$\sigma(°)$	θ(srad)
1	Truncated tetrahedron	2 hexagons + 1 triangle	$3^4 6^4$	3	12	18	8	300	1.91
2	Cuboctahedron	2 squares + 2 triangles	$3^8 4^6$	4	12	24	14	300	2.47
3	Truncated cube	2 octagons + 1 triangle	$3^8 8^6$	3	24	36	14	330	2.80
4	Truncated octahedron	1 square + 2 hexagons	$4^6 6^8$	3	24	36	14	330	3.14
5	Small rhombicuboctahedron	1 triangle + 3 squares	$3^8 4^{18}$	4	24	48	26	330	3.48
6	Snub cube[a]	4 triangles + 1 square	$3^{32} 4^6$	5	24	60	38	330	3.59
7	Icosidodecahedron	2 triangles + 2 pentagons	$3^{20} 5^{12}$	4	30	60	32	336	3.67
8	Great rhombicuboctahedron	1 square + 1 hexagon + 1 octagon	$4^{12} 6^8 8^6$	3	48	72	26	345	3.95
9	Truncated dodecahedron	1 triangle + 2 decagons	$3^{20} 10^{12}$	3	60	90	32	348	3.87
10	Truncated icosahedron	1 pentagon + 2 hexagons	$5^{12} 6^{20}$	3	60	90	32	348	4.25
11	Small rhombicosidodecahedron	1 triangle + 2 squares + 1 pentagon	$3^{20} 4^{30} 5^{12}$	4	60	120	62	348	4.44
12	Snub dodecahedron[a]	4 triangles + 1 pentagon	$3^{80} 5^{12}$	5	60	150	92	348	4.51
13	Great rhombicosidodecahedron	1 square + 1 hexagon + 1 decagon	$4^{30} 6^{20} 10^{12}$	3	120	180	62	354	4.71

[a] Chiral polyhedron

sums σ of planar angles (α) of polygons meeting at a vertex, and the solid angle θ at each vertex [24,25].

2 Results and discussion

We explored *all truncated semiregular polyhedra*, and the result is presented in Table 3. If the semiregular polyhedron before truncation has vertex degree d and V_{bt} vertices,

Table 3 Truncated semiregular polyhedra and some of their numerical data

No.	Semiregular polyhedron before truncation	New polyhedron after truncation	Symbol after truncation	V	E	F	$\sigma(^\circ)$
14	Truncated tetrahedron	Bitruncated tetrahedron	$3^{12}6^{4}12^{4}$	36	54	20	330
15	Cuboctahedron	Truncated cuboctahedron	$4^{12}6^{8}8^{6}$	48	72	26	345
16	Truncated cube	Bitruncated cube	$3^{24}6^{8}16^{6}$	72	108	38	315, 330
17	Truncated octahedron	Bitruncated octahedron	$3^{24}8^{6}12^{8}$	72	108	38	345, **360**
18	Small rhombicuboctahedron	Truncated small rhombicuboctahedron	$4^{24}6^{8}8^{18}$	96	144	50	345, **360**
19	Snub cube[a]	Truncated snub cube[a]	$5^{24}6^{32}8^{6}$	120	180	62	348, **363**
20	Icosidodecahedron	Truncated icosidodecahedron	$4^{30}6^{20}10^{12}$	120	180	62	354
21	Great rhombicuboctahedron	Truncated great rhombicuboctahedron	$3^{48}8^{12}12^{8}16^{6}$	144	216	74	345, 352.5, **367.5**
22	Truncated dodecahedron	Bitruncated dodecahedron	$3^{60}6^{20}20^{12}$	180	270	92	342, **384**
23	Truncated icosahedron	Bitruncated icosahedron	$3^{60}10^{12}12^{20}$	180	270	92	354, **360**
24	Small rhombicosidodecahedron	Truncated small rhombicosidodecahedron	$4^{60}6^{20}8^{30}10^{12}$	240	360	122	345, 354, **369**
25	Snub dodecahedron[a]	Truncated snub dodecahedron[a]	$5^{60}6^{80}10^{12}$	300	450	152	**390, 432**
26	Great rhombicosidodecahedron	Truncated great rhombicosidodecahedron	$3^{120}8^{30}12^{20}20^{12}$	360	540	182	345, 357, **372**

[a] Chiral polyhedron

then after truncation the vertex degree will be 3, and the number of vertices will be $V_{at} = d \times V_{bt}$. No longer are the resulting polyhedra vertex-transitive, so that sums of planar angles may differ from vertex to vertex. It can be seen that for some of the resulting truncated semiregular polyhedra these sums of planar angles (σ) at some of their vertices is equal to, or higher than 360°, meaning that at such vertices there is a plane or a negative curvature (concavity) of the truncated polyhedron. The names of five Archimedean polyhedra (**1, 3, 4, 9, 10**) contain the word 'truncated'; after the next truncation, the corresponding names for **14, 16, 17, 22**, and **23** should be 'bitruncated' rather than 'truncated-truncated'.

In Table 3 the nonconvex polyhedra and the σ value equal to, or higher than 360°, are shown in boldface. Four of the semiregular polyhedra afford convex polyhedra after truncation (**14, 15, 16, 23**), three have a 'flat vertex' (**17, 18, 23**), and six afford polyhedra with concave vertices/edges (**19, 21, 22, 24, 25, 26**). Interestingly, the 'flat vertex' sum of planar angles (σ) is derived in two different ways, namely:

$$90^\circ + 135^\circ + 135^\circ = 60^\circ + 150^\circ + 150^\circ = 360^\circ$$

An illustration of the truncated small rhombicuboctahedron (**18**) is provided by a stereo-view in Fig. 3. Because this stereo-view was obtained with the CambridgeSoft

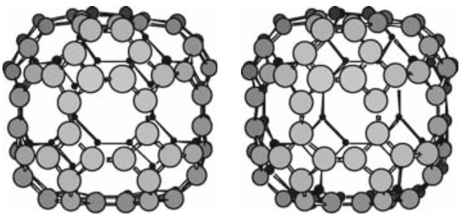

Fig. 3 Stereo-view of the truncated small rhombicuboctahedron, **18**

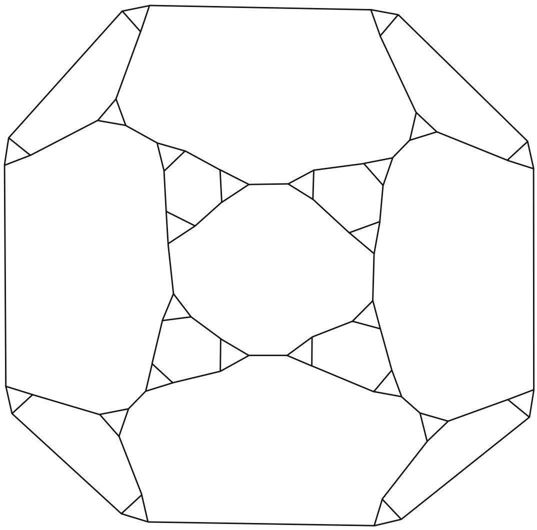

Fig. 4 Schlegel diagram of the bitruncated cube (**16**)

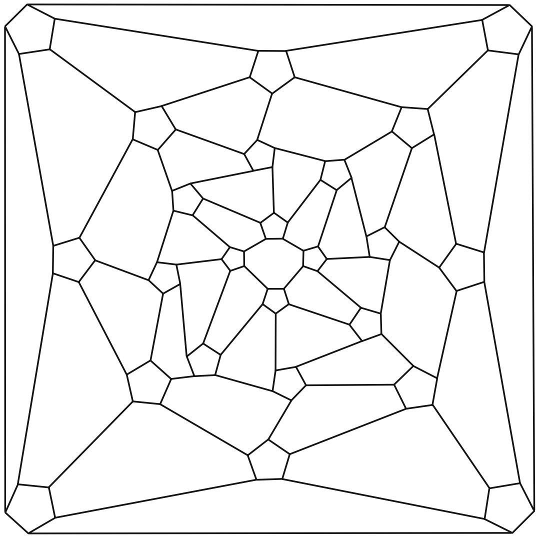

Fig. 5 Schlegel diagram of the truncated snub cube (**19**)

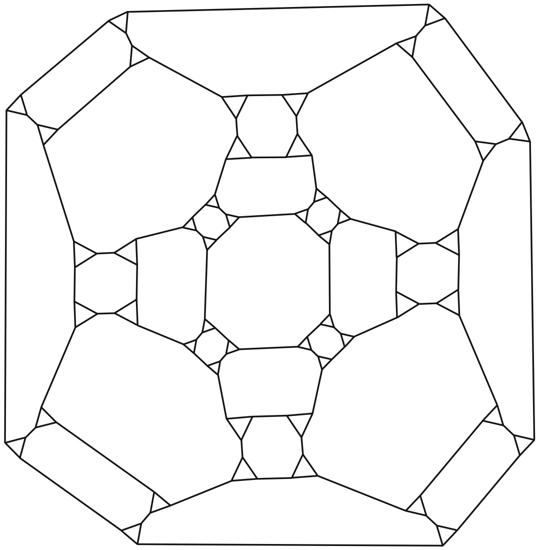

Fig. 6 Schlegel diagram of the truncated great rhombicuboctahedron (**21**)

Chem3D-Pro software (that uses Molecular Mechanics force field calculations taking into account molecular geometries), the geometry of the 'flat vertices' is distorted at the expense of steric strain.

For a few other truncated semiregular polyhedra, Schlegel diagrams provide illustrations: Fig. 4 for **16**, Fig. 5 for **19**, and Fig. 6 for **21**.

By repeating once more the truncation, all resulting polyhedra will now contain non-convex vertices because of the doubling of vertex numbers in all faces before the truncation.

It is an open question whether some of the peaks observed in the mass spectra of fullerene mixtures resulting from high-energy carbon may correspond to molecules such as the truncated snub cube (**19**).

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